On the Differentiation of inexplicable Functions *

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§367 Here, I call those functions inexplicable which cannot be defined either by determined expressions nor by means of the roots of equations such that they are not only not algebraic but it is also uncertain which kind of transcendental quantities they must be referred to. An inexplicable function of this kind is

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x},$$

which certainly depends on x, but, if x is not an integer, cannot be explained in any way. In like manner, this expression

$$1\cdot 2\cdot 3\cdot 4\cdots x$$

will be an inexplicable function of x, since, if x is any number, its value will not only be not algebraic, but even cannot be expressed by means of a certain kind of transcendental quantities. In general, the notion of such inexplicable functions can be derived from series. For, let any series be propounded

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & 4 & & \cdots & & x \\ A & + & B & + & C & + & D & + & \cdots & + & X, \end{array}$$

*Original title: " De Differentiatione Functionum inexplicabilum", first published as part of the book „*Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, 1755", reprinted in in „*Opera Omnia*: Series 1, Volume 10, pp. 588 - 618 ", Eneström-Number E212, translated by: Alexander Aycock for the „Euler-Kreis Mainz" whose sum, if it cannot be expressed by means of a finite formula, will yield an inexplicable function of x, namely,

$$S = A + B + C + D + \cdots + X.$$

Similarly, infinite products of terms of series as

$$P = A \cdot B \cdot C \cdot D \cdots X$$

will exhibit inexplicable functions of x which by means of logarithms can be reduced to the first form; for, it will be

$$\ln P = \ln A + \ln B + \ln C + \ln D + \cdots + \ln X.$$

§368 Therefore, I decided to a explain a method to investigate the differentials of inexplicable functions of this kind in this chapter. This subject, although it seems to belong to the first part of the book, where the rules of differential calculus where treated, nevertheless, since it requires a broader cognition of the doctrine of series one was only able to get to in this second part, let us, then forced to leave the natural order, treat it here now. But because this investigation is completely new and has not been done by anybody until now, in order to discuss this part of differential calculus, it is only required that we rather try to sketch its first elements here. Furthermore, I will propound several questions whose answer requires the differentiation of inexplicable functions of this kind, by means of which at the same time the use of this treatment, which without any doubt will be a lot greater in the future, is seen more clearly.

§369 To differentiate inexplicable functions of this kind it is especially necessary that we investigate their values which they have, if one substitutes $x + \omega$ for x. Therefore, let

$$\begin{array}{ccccccccccc} & & 1 & & 2 & & 3 & & 4 & \cdots & & x \\ S & = & A & + & B & + & C & + & D & + & \cdots & + & X \end{array}$$

and put Σ for the value of S which it has, if one substitutes $x + \omega$ for x, and let Z be the term of the series corresponding to the index $x + \omega$. Now, denote the terms corresponding to the indices $x + 1$, $x + 2$, $x + 3$ etc. by X', X'', X'''

etc. and the one corresponding to the infinite index $x+\infty$ by $X^{|\infty|}$. And in like manner indicate the terms corresponding to the indices $x+\omega+1$, $x+\omega+2$, $x+\omega+3$ etc. by Z', Z'', Z''' etc. and let $Z^{|\infty|}$ be the term corresponding to the index $x+\omega+\infty$. Having constituted all this it will be

$$
\begin{aligned}
S' &= S + X' \\
S'' &= S + X' + X'' \\
S''' &= S + X' + X'' + X''' \\
&\text{etc.} \\
S^{|\infty|} &= S + X' + X'' + X''' + \cdots + X^{|\infty|}
\end{aligned}
$$

Since in like manner also Σ is successively augmented by the terms Z', Z'' etc., it will be

$$
\begin{aligned}
\Sigma' &= \Sigma + Z' \\
\Sigma'' &= \Sigma + Z' + Z'' \\
\Sigma''' &= \Sigma + Z' + Z'' + Z''' \\
&\text{etc.} \\
\Sigma^{|\infty|} &= \Sigma + Z' + Z'' + Z''' + \cdots + Z^{|\infty|}
\end{aligned}
$$

§370 Now, the nature of the series S, S', S'', S''' etc. it will have, if continued to infinity, is to be considered; if the series is confounded with an arithmetic progression at infinity, what happens, if the terms of the series X, X', X'', X''' etc. converge to a ratio of 1 such that the differences of the series S, S', S'' etc. finally become equal, in this case the quantities $S^{|\infty|}$, $S^{|\infty+1|}$, $S^{|\infty+2|}$ etc. will be terms of an arithmetic progression, and because it is $\Sigma^{|\infty|}=S^{|\infty+\omega|}$, because of

$$S^{|\infty+\omega|} = S^{|\infty|} + \omega(S^{|\infty+1|} - S^{|\infty|}) = \omega S^{|\infty+1|} + (1-\omega)S^{|\infty|}$$

it will be

$$\Sigma^{|\infty|} = \omega S^{|\infty+1|} + (1-\omega)S^{|\infty|}.$$

But it is $S^{|\infty+1|} = S^{|\infty|} + X^{|\infty+1|}$, whence it is

$$\Sigma^{|\infty|} = S^{|\infty|} + \omega X^{|\infty+1|},$$

from which one will obtain this equation

$$\begin{aligned}&\Sigma + Z' + Z'' + Z''' + \cdots + Z^{|\infty|}\\ &= S + X' + X'' + X''' + \cdots + X^{|\infty|} + \omega X^{|\infty+1|},\end{aligned}$$

from which the value in question Σ the functions S has, if in it $x+\omega$ is substituted for x, will be

$$\begin{aligned}\Sigma = S + \omega X^{|\infty+1|} &+ X' + X'' + X''' + \text{etc.} \quad \text{to infinity}\\ &- Z' - Z'' - Z''' - \text{etc.} \quad \text{to infinity}\end{aligned}$$

Hence, if the infinitesimal terms of the series A, B, C, D etc. vanish, the term $\omega X^{|\infty+1|}$ vanishes and can be omitted.

§371 Therefore, the value of Σ is expressed by means of new infinite series which can be exhibited, if knows has the general term of the series $A + B + C + \text{etc.}$ from which the values of the terms Z', Z'', Z''' etc. can be defined. Therefore, having put ω to be infinitely small, since $\Sigma - S$ is the differential of the function S, this differential dS will be expressed by means of an infinite series. And if not even the higher powers of ω are neglected, one will have the complete differential of this inexplicable function S; to show its nature quite plainly, we will consider the following examples.

Example 1

To find the differential of this inexplicable function

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x}.$$

Since the general term X of this series is $= \frac{1}{x}$ and therefore

$$\begin{array}{ll|ll}
X' & = \dfrac{1}{x+1} & Z' & = \dfrac{1}{x+1+\omega} \\
X'' & = \dfrac{1}{x+2} & Z'' & = \dfrac{1}{x+2+\omega} \\
X''' & = \dfrac{1}{x+3} & Z''' & = \dfrac{1}{x+3+\omega} \\
 & \text{etc.} & & \text{etc.,}
\end{array}$$

because of

$$X^{|\infty+1|} = \frac{1}{x+\infty+1} = 0,$$

if one puts $x+\omega$ instead of x, the function S will go over into Σ that it is

$$\begin{aligned}
\Sigma = S &+ \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \text{etc.} \\
&- \frac{1}{x+1+\omega} - \frac{1}{x+2+\omega} - \frac{1}{x+3+\omega} - \text{etc.},
\end{aligned}$$

or by collecting each to terms into single ones it will be

$$\Sigma = S + \frac{\omega}{(x+1)(x+1+\omega)} + \frac{\omega}{(x+2)(x+2\omega)} + \frac{\omega}{(x+3)(x+3+\omega)} + \text{etc.},$$

or because it is

$$\begin{aligned}
\frac{1}{x+1+\omega} &= \frac{1}{x+1} - \frac{\omega}{(x+1)^2} + \frac{\omega^2}{(x+1)^3} - \frac{\omega^3}{(x+1)^4} + \text{etc.} \\
\frac{1}{x+2+\omega} &= \frac{1}{x+2} - \frac{\omega}{(x+2)^2} + \frac{\omega^2}{(x+2)^3} - \frac{\omega^3}{(x+2)^4} + \text{etc.} \\
&\text{etc.,}
\end{aligned}$$

having ordered the series according to powers of ω it will be

$$
\begin{aligned}
\Sigma = S + \omega\ &\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \text{etc.}\right)\\
- \omega^2&\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \text{etc.}\right)\\
+ \omega^3&\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \text{etc.}\right)\\
- \omega^4&\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \text{etc.}\right)\\
&\text{etc.}
\end{aligned}
$$

Having taken dx for ω we will obtain the complete differential of the propounded function S

$$
\begin{aligned}
dS = dx\ &\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \text{etc.}\right)\\
- dx^2&\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \text{etc.}\right)\\
+ dx^3&\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \text{etc.}\right)\\
- dx^4&\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \text{etc.}\right)\\
&\text{etc.}
\end{aligned}
$$

EXAMPLE 2

To find the differential of this inexplicable function of x

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2x-1}.$$

Since the general term of this series is $X = \frac{1}{2x-1}$, it will be

$$
\begin{array}{ll|ll}
X' &= \dfrac{1}{2x+1} & Z' &= \dfrac{1}{2x+1+\omega} \\
X'' &= \dfrac{1}{2x+3} & Z'' &= \dfrac{1}{2x+3+\omega} \\
X''' &= \dfrac{1}{2x+5} & Z''' &= \dfrac{1}{2x+5+\omega} \\
& \text{etc.} & & \text{etc.,}
\end{array}
$$

Because of the vanishing and equal infinitesimal terms of this series the value of S, if one puts $x+\omega$ instead of x, will result as

$$
\begin{aligned}
\Sigma = S + \frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \text{etc.} \\
- \frac{1}{2x+1+2\omega} - \frac{1}{2x+3+2\omega} - \frac{1}{2x+5+2\omega} - \text{etc.}
\end{aligned}
$$

or

$$
\Sigma = S + \frac{2\omega}{(2x+1)(2x+1+2\omega)} + \frac{2\omega}{(2x+3)(2x+3+2\omega)} + \text{etc.}
$$

But if the single terms are expanded into a power series in ω, it will be

$$
\begin{aligned}
\Sigma = S + \; & 2\omega \left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.} \right) \\
- \; & 4\omega^2 \left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.} \right) \\
+ \; & 8\omega^3 \left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \text{etc.} \right) \\
- \; & 16\omega^4 \left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \text{etc.} \right) \\
& \text{etc.}
\end{aligned}
$$

Now put dx for ω and the complete differential of the propounded inexplicable function S will be

$$
\begin{aligned}
dS = {} & 2dx \left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.} \right) \\
- {} & 4dx^2 \left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.} \right) \\
+ {} & 8dx^3 \left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \text{etc.} \right) \\
- {} & 16dx^4 \left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \text{etc.} \right) \\
& \qquad\qquad\qquad \text{etc.}
\end{aligned}
$$

EXAMPLE 3

To find the complete differential of this inexplicable function of x

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{x^n}.$$

Since the general term of this series is $= \frac{1}{x^n}$, the infinitesimal terms will be vanishing and equal to each other. And hence because of

$$
\begin{array}{ll|ll}
X' & = \dfrac{1}{(x+1)^n} & Z' & = \dfrac{1}{(x+1+\omega)^n} \\
X'' & = \dfrac{1}{(x+2)^n} & Z'' & = \dfrac{1}{(x+2+\omega)^n} \\
X''' & = \dfrac{1}{(x+3)^n} & Z''' & = \dfrac{1}{(x+3+\omega)^n} \\
& \text{etc.} & & \text{etc.,}
\end{array}
$$

it will be

$$
\begin{aligned}
X' - Z' &= \frac{n\omega}{(x+1)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+1)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+1)^{n+3}} - \text{etc.} \\
X'' - Z'' &= \frac{n\omega}{(x+2)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+2)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+2)^{n+3}} - \text{etc.} \\
& \qquad \text{etc.,}
\end{aligned}
$$

from which one finds

$$\begin{aligned}\Sigma - S = n\omega&\left(\frac{1}{(x+1)^{n+1}}+\frac{1}{(x+2)^{n+1}}+\frac{1}{(x+3)^{n+1}}+\text{etc.}\right)\\ -\frac{n(n+1)}{1\cdot 2}\omega^2&\left(\frac{1}{(x+1)^{n+2}}+\frac{1}{(x+2)^{n+2}}+\frac{1}{(x+3)^{n+2}}+\text{etc.}\right)\\ +\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\omega^3&\left(\frac{1}{(x+1)^{n+3}}+\frac{1}{(x+2)^{n+3}}+\frac{1}{(x+3)^{n+3}}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

Therefore, having put $\omega = dx$ the complete differential in question of the function S will be

$$\begin{aligned}dS = ndx&\left(\frac{1}{(x+1)^{n+1}}+\frac{1}{(x+2)^{n+1}}+\frac{1}{(x+3)^{n+1}}+\text{etc.}\right)\\ -\frac{n(n+1)}{1\cdot 2}dx^2&\left(\frac{1}{(x+1)^{n+2}}+\frac{1}{(x+2)^{n+2}}+\frac{1}{(x+3)^{n+2}}+\text{etc.}\right)\\ +\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}dx^3&\left(\frac{1}{(x+1)^{n+3}}+\frac{1}{(x+2)^{n+3}}+\frac{1}{(x+3)^{n+3}}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

§372 From these also the sums of these series can be interpolated or the values of the summatory terms can be exhibited, if the number of terms is not an integer number. For, if one puts $x = 0$, it will also be $S = 0$ and Σ will express the sum of as many terms as the number ω contains unities, even though this number ω is not an integer. So, if in the first example one puts

$$\Sigma = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\omega},$$

it will be

$$\Sigma = \frac{\omega}{1(1+\omega)} + \frac{\omega}{2(2+\omega)} + \frac{\omega}{3(3+\omega)} + \frac{\omega}{4(4+\omega)} + \text{etc.}$$

or

$$\sum = \omega\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right)$$
$$-\,\omega^2\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^2}+\text{etc.}\right)$$
$$-\,\omega^3\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}\right)$$
$$\text{etc.}$$

In the third example on the other hand it will be

$$\Sigma = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\cdots+\frac{1}{\omega^n}.$$

The value of Σ, whether ω is an integer number or a fractional number, will be expressed by means of the following series

$$\Sigma = n\omega\left(1+\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}+\frac{1}{4^{n+1}}+\text{etc.}\right)$$
$$-\frac{n(n+1)}{1\cdot 2}\omega^2\left(1+\frac{1}{2^{n+2}}+\frac{1}{3^{n+2}}+\frac{1}{4^{n+2}}+\text{etc.}\right)$$
$$+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\omega^3\left(1+\frac{1}{2^{n+3}}+\frac{1}{3^{n+3}}+\frac{1}{4^{n+3}}+\text{etc.}\right)$$
$$\text{etc.}$$

§373 These same things can also be applied to a general series; for, because it is

$$\begin{array}{cccccccccccc} & & 1 & & 2 & & 3 & & 4 & \cdots\cdots & & x \\ S & = & A & + & B & + & C & + & D & + & \cdots & + X \end{array}$$

and having put $x+\omega$ instead of x X goes over into Z and S into Σ, it will be

$$Z = X+\frac{\omega dX}{dx}+\frac{\omega^2 ddX}{1\cdot 2dx^2}+\frac{\omega^3 d^3X}{1\cdot 2\cdot 3dx^3}+\text{etc.},$$

and since in like manner Z', Z'', Z''' etc. are expressed by means of X', X'', X''' etc., it will be

$$\begin{aligned}\Sigma = S + \omega X^{|\infty+1|} - \frac{\omega}{dx} & d.(X' + X'' + X''' + X'''' + \text{etc.}) \\ -\frac{\omega^2}{1\cdot 2dx^2} & dd.(X' + X'' + X''' + X'''' + \text{etc.}) \\ -\frac{\omega^3}{1\cdot 2\cdot 3dx^3} & d^3.(X' + X'' + X''' + X'''' + \text{etc.}) \\ & \text{etc.,}\end{aligned}$$

and if $X^{|\infty+1|}$ is not $= 0$, it can be expressed in this way that the consideration of the infinity is avoided

$$X^{|\infty+1|} = X' + (X'' - X') + (X''' - X'') + (X'''' - X''') + \text{etc.}$$

and therefore it will be

$$\begin{aligned}\Sigma = S + \omega X' + \omega((X'' - X') + (X''' - X'') + (X'''' - X''') + \text{etc.}) \\ - \frac{\omega}{dx} d.(X' + X'' + X''' + X'''' + \text{etc.}) \\ - \frac{\omega^2}{2dx^2} dd.(X' + X'' + X''' + X'''' + \text{etc.}) \\ - \frac{\omega^3}{6dx^3} d^3.(X' + X'' + X''' + X'''' + \text{etc.}) \\ \text{etc.}\end{aligned}$$

If one puts $\omega = dx$, the following differential of

$$S = A + B + C + \cdots + X$$

will result expressed this way

$$\begin{aligned}dS = X'dx + dx((X'' - X') + (X''' - X'') + (X'''' - X''') + \text{etc.}) \\ -d.(X' + X'' + X''' + X'''' + \text{etc.}) \\ -\frac{1}{2} dd.(X' + X'' + X''' + X'''' + \text{etc.}) \\ -\frac{1}{6} d^3.(X' + X'' + X''' + X'''' + \text{etc.}) \\ \text{etc.}\end{aligned}$$

§374 Let us put that it is $x=0$; it will be

$$X' = A, \quad X'' = B \quad \text{etc.}$$

and hence $X' + X'' + X''' + \text{etc.}$ will be an infinite series whose general term is $= X$. Further, form the series from these general terms

$$\frac{dX}{dx}, \quad \frac{ddX}{2dx^2}, \quad \frac{d^3X}{6dx^3}, \quad \frac{d^4X}{24dx^4} \quad \text{etc.}$$

the sum of which series, if they are continued to infinity, we want to be

$$\mathcal{S}X = \mathfrak{A}, \quad \mathcal{S}\frac{dX}{dx} = \mathfrak{B}, \quad \mathcal{S}\frac{ddX}{2dx^2} = \mathfrak{C}, \quad \mathcal{S}\frac{d^3X}{6dx^3} = \mathfrak{D} \quad \text{etc.};$$

and since having put $x=0$ also $S=0$, Σ will be the sum of the series

$$A + B + C + D + \cdots + Z$$

containing ω terms; for, Z is the term of the index ω, no matter whether ω is an integer number or a fraction. Therefore, one will have

$$\begin{aligned} \Sigma = \omega A + \omega((B-A) + (C-B) + (D-C) + \text{etc.}) \\ -\omega\mathfrak{B} - \omega^2\mathfrak{C} - \omega^3\mathfrak{D} - \omega^4\mathfrak{E} - \text{etc.}, \end{aligned}$$

where the first series can be omitted, if the terms of the propounded series finally vanish.

§375 Now, let us write x instead of ω and Σ will go over into S such that it is

$$\begin{array}{rcccccc} & 1 & 2 & 3 & 4 & \cdots\cdots & x \\ S = & A + & B + & C + & D + & \cdots + & X \end{array}$$

and the same value of S will be expressed by means of an infinite series this way

$$\begin{aligned} S = Ax + x((B-A) + (C-B) + (D-C) + \text{etc.}) \\ -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \mathfrak{F}x^5 - \text{etc.}; \end{aligned}$$

since its value is expressed equally distinctly, no matter whether x is an integer number or a fraction, one is able to express the value of S of any order from this easily as:

$$\frac{dS}{dx} = A + (B - A) + (C - B) + (D - C) + \text{etc.}$$

$$\begin{aligned}
& -\mathfrak{B} - 2\mathfrak{C}x - 3\mathfrak{D}x^2 - 4\mathfrak{E}x^3 - \text{etc.} \\
\frac{ddS}{2dx^2} &= -\mathfrak{C} - 3\mathfrak{D}x - 6\mathfrak{E}x^2 - 10\mathfrak{F}x^3 - \text{etc.} \\
\frac{d^3S}{6dx^3} &= -\mathfrak{D} - 4\mathfrak{E}x - 10\mathfrak{F}x^2 - 20\mathfrak{G}x^3 - \text{etc.} \\
\frac{d^4S}{24dx^4} &= -\mathfrak{E} - 5\mathfrak{F}x - 15\mathfrak{G}x^2 - 35\mathfrak{H}x^3 - \text{etc.}
\end{aligned}$$

Therefore, since the complete differential is

$$= dS + \frac{1}{2}ddS + \frac{1}{6}d^3S + \frac{1}{24}d^4S + \text{etc.},$$

the complete differential of the propounded function S will be

$$\begin{gathered}
dS = Adx + (B - A)dx + (C - B)dx + (D - C)dx + \text{etc.} \\
-\mathfrak{B}dx - \mathfrak{C}(2xdx + dx^2) - \mathfrak{D}(3x^2dx + 3xdx^2 + dx^3) \\
-\mathfrak{E}(4x^3dx + 6x^2dx^2 + 4xdx^3 + dx^4) - \text{etc.}
\end{gathered}$$

§376 Therefore, this way the complete differential of any inexplicable function S can be assigned, if the infinitesimal terms of the series

$$A + B + C + D + \text{etc.}$$

either vanish or become equal to each other. For, if the infinitesimal terms of this series were not $= 0$, then the sum of the series $\mathfrak{B}$ which is formed from the general term $\frac{dX}{dx}$, will become infinite, but together with the series

$$A + (B - A) + (C - B) + (D - C) + \text{etc.}$$

it will constitute a finite sum. But it can happen that the terms of the series $A + B + C + D+$etc. are augmented to infinity in such a way that not only the sum of the series $\mathfrak{B}$, but also the sum of the series $\mathfrak{C}$ becomes infinitely large, in which case it does not suffice to have added the series $A+(B-A)+(C-B)+$etc.; but since in this case the infinitesimal values considered in § 370, namely $S^{|\infty|}$, $S^{|\infty+1|}$, $S^{|\infty+2|}$, are not any longer terms in an arithmetic progression, as we had assumed before, the nature of this progression will have to be taken into account. As we assumed the first differences of these progressions to be equal, so we will extend the method even further, if we set that just the second or the third or the higher differences of these values become constant.

§377 Arguing exactly as before in § 369, let us put that just the second differences of the mentioned values are constant

$$S^{|\infty|}, \quad S^{|\infty+1|}, \quad S^{|\infty+2|}$$

First Differences

$$X^{|\infty+1|}, \quad X^{|\infty+2|}$$

Second Differences

$$X^{|\infty+2|} - X^{|\infty+1|}$$

Therefore, it will be

$$\begin{aligned}\Sigma^{|\infty|} = S^{|\infty+\omega|} &= S^{|\infty|} + \omega X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2}(X^{|\infty+2|} - X^{|\infty+1|}) \\ &= S^{|\infty|} - \frac{\omega(\omega-3)}{1\cdot 2}X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2}X^{|\infty+2|}.\end{aligned}$$

Therefore, one will have this equation

$$\begin{gathered}\Sigma + Z' + Z'' + Z''' + \cdots + Z^{|\infty|} \\ = S + X' + X'' + X''' + \cdots + X^{|\infty|} - \frac{\omega(\omega-3)}{1\cdot 2}X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2}X^{|\infty+2|},\end{gathered}$$

from which one finds

$$\begin{aligned}\Sigma = S &+ X' + X'' + X''' + X'''' + \text{etc. to infinity} \\ &- Z' - Z'' - Z''' - Z'''' - \text{etc. to infinity} \\ &+ \omega X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2}(X^{|\infty+2|} - X^{|\infty+1|}).\end{aligned}$$

But these infinitesimal terms can be represented in such a way that it is

$$\begin{aligned}\Sigma = S + X' + X'' + X''' + X'''' + \text{etc. to infinity}\\ - Z' - Z'' - Z''' - Z'''' - \text{etc. to infinity}\end{aligned}$$

$$+\omega X' + \omega \left\{ \begin{array}{l} + X'' + X''' + X'''' + X''''' + \text{etc.} \\ - X' - X'' - X''' + X'''' - \text{etc.} \end{array} \right\}$$

whence the law describing the nature of this expression, if just the third or fourth or higher differences were constant, is obvious.

§378 Because it is, as we demonstrated above,

$$Z = X + \frac{\omega dX}{1dx} + \frac{\omega^2 ddX}{1 \cdot 2dx^2} + \frac{\omega^3 d^3 X}{1 \cdot 2 \cdot 3dx^3} + \text{etc.},$$

if for Z', Z'', Z''' etc. we substitute the values to result from them, the value of S, if one writes $x + \omega$ instead of x, will be the following:

$$\Sigma = S + \omega X' + \omega \left\{ \begin{array}{l} + X'' + X''' + X'''' + X''''' + \text{etc.} \\ - X' - X'' - X''' + X'''' - \text{etc.} \end{array} \right\}$$

$$\begin{array}{l} + \dfrac{\omega(\omega-1)}{1 \cdot 2} X'' \\ - \dfrac{\omega(\omega-1)}{1 \cdot 2} X' \end{array} + \frac{\omega(\omega-1)}{1 \cdot 2} \left\{ \begin{array}{l} + X''' + X'''' + X''''' + \text{etc.} \\ - 2X'' - 2X''' - 2X'''' - \text{etc.} \\ + X' + X'' + X''' + \text{etc.} \end{array} \right\}$$

$$\begin{aligned} &- \frac{\omega}{dx} d.\ (X' + X'' + X''' + X'''' + \text{etc.}) \\ &- \frac{\omega^2}{2dx^2} d^2.(X' + X'' + X''' + X'''' + \text{etc.}) \\ &- \frac{\omega^3}{6dx^3} d^3.(X' + X'' + X''' + X'''' + \text{etc.}) \\ &\qquad \text{etc.} \end{aligned}$$

If one puts dx instead of ω, the complete differential of the propounded inexplicable function S will result, namely

$$dS = X'dx + dx \left\{ \begin{array}{l} + X'' + X''' + X'''' + X''''' + \text{etc.} \\ - X' \;- X'' \;- X''' \;+ X'''' \;- \text{etc.} \end{array} \right\}$$

$$\begin{array}{l} - X'' \dfrac{dx(1-dx)}{1\cdot 2} \\ + X' \dfrac{dx(1-dx)}{1\cdot 2} \end{array} - \frac{dx(1-dx)}{1\cdot 2} \left\{ \begin{array}{l} + \;X''' + \;X'''' + \;X''''' + \text{etc.} \\ - 2X'' - 2X''' - 2X'''' - \text{etc.} \\ + \quad X' + \quad X'' + \quad X''' + \text{etc.} \end{array} \right\}$$

$$\begin{array}{l} + X''' \dfrac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3} \\ - 2X'' \dfrac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3} \\ + X' \quad \dfrac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3} \end{array} - \frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3} \left\{ \begin{array}{l} + \;X'''' + \;X''''' + \text{etc.} \\ - 3X''' - 3X'''' - \text{etc.} \\ + \;3X'' + \;3X''' + \text{etc.} \\ - \quad X'' - \quad X''' - \text{etc.} \end{array} \right\}$$

etc.

$$\begin{aligned} &-d.(X' + X'' + X''' + X'''' + X''''' + \text{etc.}) \\ -&\frac{1}{2}dd.(X' + X'' + X''' + X'''' + X''''' + \text{etc.}) \\ -&\frac{1}{6}d^3.(X' + X'' + X''' + X'''' + X''''' + \text{etc.}) \\ -&\frac{1}{24}d^4.(X' + X'' + X''' + X'''' + X''''' + \text{etc.}) \end{aligned}$$

etc.

which expression extends very far and, no matter at which point the differences just became constant, will exhibit the differential in question. For, this formula is accommodated to constant differences and at the same time the law is plain, if it is necessary to proceed further.

§379 If the series $A+B+C+D+$ etc., from which the inexplicable function

$$\begin{array}{ccccccccccc} & & 1 & & 2 & & 3 & & 4 & & & & x \\ S & = & A & + & B & + & C & + & D & + & \cdots & + & X \end{array}$$

is formed, was of such a nature that the infinitesimal terms vanish, then, as we already noted, it will be

$$\begin{aligned} dS = & -d.(X' + X'' + X''' + X'''' + \text{etc.}) \\ & -\frac{1}{2}dd.(X' + X'' + X''' + X'''' + \text{etc.}) \\ & -\frac{1}{6}d^3(X' + X'' + X''' + X'''' + \text{etc.}) \\ & -\frac{1}{24}d^4.(X' + X'' + X''' + X'''' + \text{etc.}) \end{aligned}$$

But if the infinitesimal terms of that series were not $=0$, but nevertheless have vanishing differences, then additionally this expression is to be added

$$dx \left\{ \begin{array}{lllllll} & + X'' & + X''' & + X'''' & + X''''' & + \text{etc.} \\ X' & & & & & \\ & - X' & - X'' & - X''' & + X'''' & - \text{etc.} \end{array} \right\}$$

But if just the second differences of the infinitesimal terms of this series $A+B+C+D+$ etc. vanish, then one furthermore has to add

$$\frac{dx(dx-1)}{1\cdot 2} \left\{ \begin{array}{llllll} & + X''' & + X'''' & + X''''' & + \text{etc.} \\ + X'' & & & & \\ & - 2X'' & - 2X''' & - 2X'''' & - \text{etc.} \\ - X' & & & & \\ & + X' & + X'' & + X''' & + \text{etc.} \end{array} \right\}$$

And if just the third differences of the mentioned infinitesimal terms vanish, then except for the already exhibited expressions one additionally has to add

$$\frac{dx(dx-1)(dx-2)}{1\cdot 2\cdot 3}\left\{\begin{array}{lllll} & +\ X'''' & +\ X''''' & +\ X'''''' & +\text{ etc.}\\ +\ X''' & & & & \\ & -\ 3X''' & -\ 3X'''' & -\ 3X''''' & -\text{ etc.}\\ -\ 2X'' & & & & \\ & +\ 3X'' & +\ 3X''' & +\ 3X'''' & +\text{ etc.}\\ +\ \ X' & & & & \\ & -\ \ X'' & -\ \ X''' & -\ \ X''' & -\text{ etc.}\end{array}\right\}$$

And this will be the nature of the expressions additionally to be added, if just higher differences of the infinitesimal terms of the series $A+B+C+D+$ etc. vanish. And hence, no matter which series is assumed, as long as its infinitesimal terms are finally reduced to vanishing differences, one will be able to define the differential of the inexplicable function formed from it.

§380 If one puts $x=0$, it will be $X'=A$, $X''=B$, $X'''=C$ etc. Therefore, as $A+B+C+D+$ etc. is the series whose general term is X, if from the general terms

$$\frac{dX}{dx}, \quad \frac{ddX}{2dx^2}, \quad \frac{d^3X}{6dx^3}, \quad \frac{d^4X}{24dx^4} \quad \text{etc.}$$

in like manner infinite series are formed and its sums are denoted by $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc., respectively, the sum of ω terms of the series

$$A+B+C+D+\text{etc.}$$

will be expressed in such a way that it does not matter whether ω is an integer or not. Therefore, let us put x for ω that it is

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & & & x\\ S & =A & +B & +C & +D & + & \cdots & +X\end{array}$$

and if the infinitesimal terms of this series vanish, it will be

$$S=-\mathfrak{B}x-\mathfrak{C}x^2-\mathfrak{D}x^3-\mathfrak{E}x^4-\text{etc.}$$

But if at least the infinitesimal terms have constant first differences, then one additionally has to add this

$$x\left\{\begin{array}{l} \quad + B + C + D + E + \text{etc.} \\ A \\ \quad - A - B - C + D - \text{etc.} \end{array}\right\}$$

But if just the second differences of those infinitesimal terms vanish, then furthermore one has to add

$$\frac{x(x-1)}{1\cdot 2}\left\{\begin{array}{l} \qquad + \ C + \ D + \ E + \ F + \text{etc.} \\ + B \\ \qquad - 2B - 2C - 2D - 2E - \text{etc.} \\ - C \\ \qquad + \ A + \ B + \ C + \ D + \text{etc.} \end{array}\right\}$$

If just the third differences vanish, then additionally this infinite series has to be added

$$\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}\left\{\begin{array}{l} \qquad + \ D + \ E + \ F + \ G + \text{etc.} \\ + \ C \\ \qquad - 3C - 3D - 3E - 3F - \text{etc.} \\ - 2B \\ \qquad + 3B + 3C + 3D + 3E + \text{etc.} \\ + \ A \\ \qquad - \ A - \ B - \ C - \ D - \text{etc.} \end{array}\right\}$$

etc.

§381 Let us also apply these things to the kind of inexplicable functions that consists of continuous products of several terms of the propounded series $A+B+C+D$ etc., and let

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 \cdots & x \\ S = & A \cdot & B \cdot & C \cdot & D \cdots & X \end{array}$$

and at first find the value Σ into which S is transformed, if one writes $x+\omega$ instead of x. But let us, as before, put that Z is the term corresponding to the index is $= x+\omega$ of the series $A+B+C+D+$etc. while X corresponds to the index x. To reduce this to the preceding case, let us take logarithms and it will be

$$\ln S = \ln A + \ln B + \ln C + \ln D + \cdots + \ln X.$$

If now the infinitesimal terms of this series vanish by applying the same method we used before it will be

$$\begin{aligned} \ln \Sigma = \ln S + \ln X' + \ln X'' + \ln X''' + \text{etc.} \\ - \ln Z' - \ln Z'' - \ln Z''' - \text{etc.} \end{aligned}$$

and hence by going back to numbers it will be

$$\Sigma = S \cdot \frac{X'}{Z'} \cdot \frac{X''}{Z''} \cdot \frac{X'''}{Z'''} \cdot \frac{X''''}{Z''''} \cdot \text{etc.};$$

therefore, this expression holds, if the infinitesimal terms of the series A, B, C, D etc. become equal to 1. But if the logarithms of the infinitesimal terms of this series do not vanish, but nevertheless have vanishing differences, then to that series we found for $\ln \Sigma$ one additionally has to add this series

$$\omega \ln X' + \omega \left(\ln \frac{X''}{X'} + \ln \frac{X'''}{X''} + \ln \frac{X''''}{X'''} + \text{etc.} \right)$$

and so by taking numbers one will have

$$\Sigma = S \cdot X'^{\omega} \cdot \frac{X'^{\omega} X''^{1-\omega}}{Z'} \cdot \frac{X'''^{\omega} X''^{1-\omega}}{Z''} \cdot \frac{X''''^{\omega} X'''^{1-\omega}}{Z'''} \cdot \text{etc.}$$

§382 If we put $x = 0$ in which case $S = 1$ and $X' = A$, $X'' = B$, $X''' = C$ etc., Σ will denote the product of ω terms of this series A, B, C, D etc. If we write x for ω that Σ obtains the value we had attributed to S before such that it is

$$\begin{aligned} &\quad\ 1 \quad 2 \quad 3 \quad 4 \cdots x \\ S &= A \cdot B \cdot C \cdot D \cdots X, \end{aligned}$$

since now Z', Z'', Z''' etc. go over into X', X'', X''' etc., if the logarithms of the infinitesimal terms of this series A, B, C, D, E etc. vanish, S will be expressed this way

$$S=\frac{A}{X'}\cdot\frac{B}{X''}\cdot\frac{C}{X'''}\cdot\frac{D}{X''''}\cdot\frac{E}{X'''''}\cdot\text{etc.}$$

But if just the differences of the logarithms of the infinitesimal terms of the series A, B, C, D etc. vanish, then this function S will be expressed the following way that it is

$$S=A^x\cdot\frac{B^xA^{1-x}}{X'}\cdot\frac{C^xB^{1-x}}{X''}\cdot\frac{D^xC^{1-x}}{X'''}\cdot\frac{E^xD^{1-x}}{X'''}\cdot\text{etc.};$$

If just the second differences of those logarithms vanish, it is easily concluded from the preceding, factors of which kind are to be added; we omit this case here, since it usually does not occur. Moreover, I will show the use of these expressions in the task of interpolation in the following chapter.

§383 Since here mainly the differentiation of inexplicable functions is propounded, let us investigate the differential of this function

$$S=A\cdot B\cdot C\cdot D\cdots X.$$

For this, let us go back to the equation found before

$$\begin{aligned}\ln\Sigma=\ln S+\ln X'+\ln X''+\ln X'''+\text{etc.}\\ -\ln Z'-\ln Z''-\ln Z'''-\text{etc.},\end{aligned}$$

and since $\ln Z$ results from $\ln X$, if one writes $x+\omega$ instead of x, it will be

$$\ln Z=\ln X+\frac{\omega}{dx}d.\ln X+\frac{\omega^2}{2dx^2}dd.\ln X+\frac{\omega^3}{6dx^3}d^3.\ln X+\text{etc.};$$

having substituted these values for $\ln Z'$, $\ln Z'''$, $\ln Z'''$ etc. one will have

$$\begin{aligned}\ln\Sigma=\ln S-&\frac{\omega}{dx}d.\ (\ln X'+\ln X''+\ln X'''+\ln X''''+\text{etc.})\\ -&\frac{\omega^2}{2dx^2}dd.(\ln X'+\ln X''+\ln X'''+\ln X''''+\text{etc.})\end{aligned}$$

$$-\frac{\omega^3}{6dx^3}d^3.\ (\ln X' + \ln X'' + \ln X''' + \ln X'''' + \text{etc.})$$
$$\text{etc.}$$

Now put $\omega = dx$ and it will be $\ln \Sigma = \ln S + d.\ln S$ and hence it will be

$$\begin{aligned}\frac{dS}{S} &= -d.\ (\ln X' + \ln X'' + \ln X''' + \ln X'''' + \text{etc.})\\ &-\frac{1}{2}dd.(\ln X' + \ln X'' + \ln X''' + \ln X'''' + \text{etc.})\\ &-\frac{1}{6}d^3.(\ln X' + \ln X'' + \ln X''' + \ln X'''' + \text{etc.})\\ &\text{etc.},\end{aligned}$$

which formula holds, if the logarithms of the infinitesimal terms of the series A, B, C, D etc. vanish; but if they do not vanish, but nevertheless have vanishing differences, then to the preceding expression of the complete differential one additionally has to add this series

$$dx \ln X' + dx\left(\ln\frac{X''}{X'} + \ln\frac{X'''}{X''} + \ln\frac{X''''}{X'''} + \text{etc.}\right)$$

in order to obtain the complete differential.

§384 The same can also be achieved in another way. Put $x = 0$ in which case $\ln S$ goes over into 0. Then, form series whose general terms are

$$\ln X, \quad \frac{d.\ln X}{dx}, \quad \frac{dd.\ln X}{2dx^2}, \quad \frac{d^3.\ln X}{6dx^3} \quad \text{etc.},$$

and we want the sums of these infinite series to be $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., respectively. Write x for ω that $\Sigma = S$ and it will be

$$\ln S = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \text{etc.},$$

if the logarithms of the infinitesimal terms of the series A, B, C, D etc. whose general term is X vanish; but if just the differences of these logarithms vanish, it will be

$$\ln S = x \ln A + x\left(\ln\frac{B}{A} + \ln\frac{C}{B} + \ln\frac{D}{C} + \ln\frac{E}{D} + \text{etc.}\right)$$

$$-\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{D}x^4 - \text{etc.}$$

And hence the differential of $\ln S$ will be

$$\frac{dS}{S} = dx \ln A + dx\left(\ln\frac{B}{A} + \ln\frac{C}{B} + \ln\frac{D}{C} + \ln\frac{E}{D} + \text{etc.}\right)$$
$$-\mathfrak{B}xdx - 2\mathfrak{C}xdx - 3\mathfrak{D}x^2dx - 4\mathfrak{E}x^3dx - \text{etc.}$$

But if the complete differential is desired, it will be

$$\frac{dS}{S} = dx \ln A + dx\left(\ln\frac{B}{A} + \ln\frac{C}{B} + \ln\frac{D}{C} + \ln\frac{E}{D} + \text{etc.}\right)$$
$$-\mathfrak{B}dx - \mathfrak{C}(2xdx + dx^2) - \mathfrak{D}(3xxdx + 3xdx^2 + dx^3) - \text{etc.}$$

To show the use of these formulas we add the following examples which we resolve in both ways.

Example 1

To find the differential of this inexplicable function

$$S = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{2x-1}{2x}.$$

Here, it is especially to be noted that the infinitesimal terms of these factors go over into 1 and hence their logarithms vanish. Since it is $X = \frac{2x-1}{2x}$, it will be

$$X' = \frac{2x+1}{2x+1}, \quad X'' = \frac{2x+3}{2x+4}, \quad X''' = \frac{2x+5}{2x+6} \quad \text{etc.}$$

and in general

$$X^{|n|} = \frac{2x+2n-1}{2x+2n};$$

therefore, it will be

$$\begin{aligned} \ln X^{|n|} &= +\ln(2x+2n-1) - \ln(2x+2n) \\ d.\ln X^{|n|} &= +\frac{2dx}{2x+2n-1} - \frac{2dx}{2x+2n} \end{aligned}$$

$$
\begin{aligned}
dd.\ln X^{|n|} &= -\frac{4dx^2}{(2x+2n-1)^2} + \frac{4dx^2}{(2x+2n)^2}\\
d^3.\ln X^{|n|} &= +\frac{2\cdot 2\cdot 4dx^2}{(2x+2n-1)^3} - \frac{2\cdot 2\cdot 4dx^2}{(2x+2n)^3}\\
d^4.\ln X^{|n|} &= -\frac{2\cdot 2\cdot 4\cdot 6dx^4}{(2x+2n-1)^4} + \frac{2\cdot 2\cdot 4\cdot 6dx^4}{(2x+2n)^4}\\
&\qquad\qquad\text{etc.;}
\end{aligned}
$$

therefore, the complete differential will be

$$
\begin{aligned}
\frac{dS}{S} = &-2dx \left\{ \begin{aligned} &\frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \text{etc.}\\ -&\frac{1}{2x+2} - \frac{1}{2x+4} - \frac{1}{2x+6} - \text{etc.} \end{aligned} \right\}\\
&+\frac{4}{2}dx^2 \left\{ \begin{aligned} &\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.}\\ -&\frac{1}{(2x+2)^2} - \frac{1}{(2x+4)^2} - \frac{1}{(2x+6)^2} - \text{etc.} \end{aligned} \right\}\\
&-\frac{8}{3}dx^3 \left\{ \begin{aligned} &\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.}\\ -&\frac{1}{(2x+2)^3} - \frac{1}{(2x+4)^3} - \frac{1}{(2x+6)^3} - \text{etc.} \end{aligned} \right\}\\
&\qquad\text{etc.}
\end{aligned}
$$

But if only the first differential is in question, it will be

$$
\frac{dS}{S} = -2dx\cdot\left(\frac{1}{(2x+1)(2x+2)} + \frac{1}{(2x+3)(2x+4)} + \frac{1}{(2x+5)(2x+6)} + \text{etc.}\right),
$$

which same is found by means of the other method given in § 394. Since it is

$$
\ln X = \ln\frac{2x-1}{2x},
$$

it will be

$$
\frac{d.\ln X}{dx} = \frac{2}{2x-1} - \frac{1}{x},\quad \frac{dd.\ln X}{2dx^2} = -\frac{2}{(2x-1)^2} + \frac{1}{2xx},
$$

$$
\frac{d^3\ln X}{6dx^3} = \frac{8}{3(2x-1)^3} - \frac{1}{3x^3}\quad \text{etc.}
$$

and hence it will be

$$\mathfrak{A} = \ln\frac{1}{2} + \ln\frac{3}{4} + \ln\frac{5}{6} + \ln\frac{7}{8} + \text{etc.}$$

$$\mathfrak{B} = \left\{\begin{array}{l} \frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \text{etc.} \\ -\frac{2}{2} - \frac{2}{4} - \frac{2}{6} - \frac{2}{8} - \frac{2}{10} - \text{etc.} \end{array}\right\} = 2\ln 2$$

$$\mathfrak{C} = -\frac{4}{2}\left\{\begin{array}{l} \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} \\ -\frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \text{etc.} \end{array}\right\}$$

$$\mathfrak{D} = +\frac{8}{3}\left\{\begin{array}{l} \frac{1}{1} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.} \\ -\frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{6^3} - \frac{1}{8^3} - \text{etc.} \end{array}\right\}$$

$$\mathfrak{E} = -\frac{16}{4}\left\{\begin{array}{l} \frac{1}{1} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} \\ -\frac{1}{2^4} - \frac{1}{4^4} - \frac{1}{6^4} - \frac{1}{8^4} - \text{etc.} \end{array}\right\}$$

$$\text{etc.}$$

or it will be

$$\mathfrak{B} = +\frac{2}{1}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.}\right)$$

$$\mathfrak{C} = -\frac{4}{2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \text{etc.}\right)$$

$$\mathfrak{D} = +\frac{8}{3}\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \text{etc.}\right)$$

$$\mathfrak{E} = -\frac{16}{4}\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \text{etc.}\right)$$

$$\text{etc.}$$

Having substituted the found values it will be

$$\begin{aligned}
\frac{dS}{S} = -2dx & \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.}\right)\\
+ \ 4xdx & \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \text{etc.}\right)\\
- \ 8x^2dx & \left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \text{etc.}\right)\\
+ 16x^3dx & \left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \text{etc.}\right)\\
& \text{etc.}
\end{aligned}$$

If $x = 0$ in which case $\ln S = 0$ and $S = 1$, it will be $dS = -2dx \ln 2$.

EXAMPLE 2

To find the differential of this inexplicable function

$$S = 1 \cdot 2 \cdot 3 \cdot 4 \cdots x.$$

The terms of this series 1, 2, 3, 4 etc. grow to infinity in such a way that the differences of the logarithms vanish; for, it is

$$\ln(\infty + 1) - \ln \infty = \ln \left(1 + \frac{1}{\infty}\right) = \frac{1}{\infty} = 0.$$

Since it is $X = x$, it will be

$$X' = x + 1, \quad X'' = x + 2, \quad X''' = x + 3 \quad \text{etc.};$$

but, further because of $\ln X = \ln x$ it will be

$$d. \ln X = \frac{dx}{x}, \quad dd. \ln X = -\frac{dx^2}{x^2}, \quad d^3. \ln X = \frac{2dx^3}{x^3}, \quad d^4. \ln X = -\frac{2 \cdot 3dx^4}{x^4} \quad \text{etc.};$$

hence, if the last logarithms would vanish, it would be

$$
\begin{aligned}
\frac{dS}{S} = -dx&\left(\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\frac{1}{x+4}+\text{etc.}\right)\\
+\frac{dx^2}{2}&\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\frac{1}{(x+4)^2}+\text{etc.}\right)\\
-\frac{dx^3}{3}&\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\frac{1}{(x+4)^3}+\text{etc.}\right)\\
&\text{etc.}
\end{aligned}
$$

But because just the differences of the logarithms vanish, one additionally has to add this expression

$$
dx\ln(x+1)+dx\left(\ln\frac{x+2}{x+1}+\ln\frac{x+3}{x+2}+\ln\frac{x+4}{x+3}+\ln\frac{x+5}{x+4}+\text{etc.}\right).
$$

But because it is

$$
\begin{aligned}
\ln\frac{x+2}{x+1} &= \frac{1}{x+1}-\frac{1}{2(x+1)^2}+\frac{1}{3(x+1)^3}-\frac{1}{4(x+1)^4}+\text{etc.}\\
\ln\frac{x+3}{x+2} &= \frac{1}{x+2}-\frac{1}{2(x+2)^2}+\frac{1}{3(x+2)^3}-\frac{1}{4(x+2)^4}+\text{etc.}\\
&\text{etc.},
\end{aligned}
$$

the complete differential will be

$$
\begin{aligned}
\frac{dS}{S} = dx\ln(x+1) &- \frac{1}{2}(dx-dx^2)\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\text{etc.}\right)\\
&+\frac{1}{3}(dx-dx^3)\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\text{etc.}\right)\\
&-\frac{1}{4}(dx-dx^4)\left(\frac{1}{(x+1)^4}+\frac{1}{(x+2)^4}+\frac{1}{(x+3)^4}+\text{etc.}\right)\\
&-\frac{1}{5}(dx-dx^5)\left(\frac{1}{(x+1)^5}+\frac{1}{(x+2)^5}+\frac{1}{(x+3)^5}+\text{etc.}\right)\\
&\text{etc.}
\end{aligned}
$$

But if we want to express this differential by means of the other method, since it is

$$\ln X = \ln x, \quad \frac{d.\ln X}{dx} = \frac{1}{x}, \quad \frac{dd.\ln X}{2dx^2} = -\frac{1}{2x^2}, \quad \frac{d^3.\ln X}{6dx^3} = \frac{1}{3x^3}, \quad \frac{d^4.\ln X}{24dx^4} = -\frac{1}{4x^4} \quad \text{etc.,}$$

one will have the following series

$$\begin{aligned}
\mathfrak{A} &= \qquad \ln 1 + \ln 2 + \ln 3 + \ln 4 + \ln 5 + \text{etc.} \\
\mathfrak{B} &= +1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.}\right) \\
\mathfrak{C} &= -\frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.}\right) \\
\mathfrak{C} &= +\frac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.}\right) \\
\mathfrak{D} &= -\frac{1}{4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.}\right) \\
& \qquad \text{etc.}
\end{aligned}$$

Therefore, because of $\ln A = \ln 1 = 0$ from § 384 it will be

$$\begin{aligned}
\ln S = x & \left(\ln\frac{2}{1} + \ln\frac{3}{2} + \ln\frac{4}{3} + \ln\frac{5}{4} + \text{etc.}\right) \\
-x & \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}\right) \\
+\frac{1}{2}x^2 & \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.}\right) \\
-\frac{1}{3}x^3 & \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.}\right) \\
+\frac{1}{4}x^4 & \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.}\right) \\
& \qquad \text{etc.}
\end{aligned}$$

But the two first series by which x is multiplied, even though both have an infinite sum, nevertheless taken together have a finite sum. For, if n terms are taken of both of them, it will be

$$\ln(n+1) - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n}.$$

But above (§ 142) we found that it is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \text{Const.} + \ln n + \frac{1}{2n} - \frac{\mathfrak{A}}{2n^2} + \frac{\mathfrak{B}}{4n^4} - \text{etc.}$$

and this constant will be found to be $= 0.5772156649015325$. If one puts $n = \infty$, it will be

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{\infty} = \text{Const.} + \ln \infty,$$

whence the value of those two series continued to infinity will be

$$= \ln(\infty + 1) - \text{Const.} - \ln \infty = -\text{Const.}$$

From this it will be

$$\begin{gathered}
\ln S = -x \cdot 0.5772156649015325 \\
+ \frac{1}{2}xx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.}\right) \\
- \frac{1}{3}x^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.}\right) \\
+ \frac{1}{4}x^4\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.}\right) \\
\text{etc.,}
\end{gathered}$$

whence the differentials of any order are easily found. For, it will be

$$\begin{gathered}
\frac{dS}{S} = -dx \cdot 0.5772156649015325 \\
+ xdx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.}\right) \\
- x^2dx\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.}\right) \\
+ x^3dx\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.}\right) \\
\text{etc.}
\end{gathered}$$

But if these series are collected into one sum, it will be

$$\frac{dS}{S} = -dx \cdot 0.5772156649015325 + \frac{xdx}{1(1+x)} + \frac{xdx}{2(2+x)} + \frac{xdx}{3(3+x)} + \frac{xdx}{4(4+x)} + \text{etc.}$$

Hence, if $x = 0$, it will be

$$\frac{dS}{S} = -dx \cdot 0.5772156649015325.$$

From the first expression on the other hand it will be in this case

$$\begin{aligned}\frac{dS}{S} = & -\frac{1}{2}dx\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}\right)\\ & +\frac{1}{3}dx\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.}\right)\\ & -\frac{1}{4}dx\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}\right)\\ & +\frac{1}{5}dx\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.}\right)\\ & \qquad \text{etc.}\end{aligned}$$

§385 Hence one is also able to exhibit the differentials of inexplicable functions of this kind in any special case, since here we found the complete differentials. Therefore, if such functions enter expressions which seem to be undetermined of which kind we treated some in the preceding chapter, one will be able to define the values by means of the same method, as it will be understood from the added examples.

Example 1

To determine the value of this expression

$$\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}}{x(x-1)} - \frac{1}{(x-1)(2x-1)}.$$

in the case, in which one puts $x = 1$.

Let us put

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}=S;$$

from § 372 it will be

$$\begin{aligned} S = x &\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}\right) \\ -x^2&\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.}\right) \\ +x^3&\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}\right) \\ &\text{etc.,} \end{aligned}$$

or because it also is

$$\begin{aligned} S = &+1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.} \\ &-\frac{1}{1+x}-\frac{1}{2+x}-\frac{1}{3+x}-\frac{1}{4+x}-\frac{1}{5+x}-\text{etc.,} \end{aligned}$$

if each term of the upper series is combined with the preceding of the lower series, it will be

$$S=1+\frac{x-1}{2(1+x)}+\frac{x-1}{3(2+x)}+\frac{x-1}{4(3+x)}+\text{etc.,}$$

which expression, since one has to put $x=1$, is more convenient. Therefore, let $x=1+\omega$ and it will be

$$S=1+\frac{\omega}{2(2+\omega)}+\frac{\omega}{3(3+\omega)}+\frac{\omega}{4(4+\omega)}+\text{etc.}$$

or

$$\begin{aligned} S = 1+\omega &\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}\right) = 1+\mathfrak{B}\omega \\ -\omega^2&\left(\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}\right) \qquad +\mathfrak{C}\omega^2 \\ -\omega^3&\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}\right) \qquad +\mathfrak{D}\omega^3 \\ &\text{etc.} \qquad\qquad\qquad \text{etc.} \end{aligned}$$

Therefore, the total expression having put $x = 1 + \omega$ will go over into

$$\frac{1 + \mathfrak{B}\omega - \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 - \text{etc.}}{\omega(1+\omega)} - \frac{1}{\omega(1+2\omega)}$$

or

$$\frac{\omega + \mathfrak{B}\omega + 2\mathfrak{B}\omega^2 - \mathfrak{C}\omega^2 - \text{etc.}}{\omega(1+\omega)(1+2\omega)} = \frac{1 + \mathfrak{B} + 2\mathfrak{B}\omega - \mathfrak{C}\omega - \text{etc.}}{(1+\omega)(1+2\omega)}.$$

Now put $\omega = 0$ and the propounded value of the expression in the case $x = 1$ will be

$$= 1 + \mathfrak{B} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.};$$

since this series is $= \frac{1}{6}\pi^2$, it follows that the value in question is $= \frac{1}{6}\pi^2$.

EXAMPLE 2

To find the value of this expression

$$\frac{2x - xx}{(x-1)^2} + \frac{\pi\pi x}{6(x-1)} - \frac{(2x-1)(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x})}{x(x-1)^2}$$

in the case $x = 1$.

Put $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x} = S$ and set $x = 1 + \omega$; it will, as we found in the preceding example, be

$$S = 1 + \mathfrak{B}\omega - \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 - \text{etc.}$$

while

$$\begin{aligned}
\mathfrak{B} &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = \frac{1}{6}\pi\pi - 1 \\
\mathfrak{C} &= \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} \\
\mathfrak{D} &= \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} \\
&\text{etc.}
\end{aligned}$$

Therefore, having put $x = 1 + \omega$ the propounded expression will take on this form

$$\frac{1-\omega\omega}{\omega\omega}+\frac{(1+\mathfrak{B})(1+\omega)}{\omega}-\frac{(1+2\omega)(1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\text{etc.})}{(1+\omega)\omega^2},$$

which reduced to the common denominator $\omega^2(1+\omega)$ becomes

$$\frac{1+\omega-\omega^2-\omega^3+2\omega^2+\omega^3+\mathfrak{B}\omega(1+2\omega+\omega\omega)-1-\mathfrak{B}\omega+\mathfrak{C}\omega^2-\mathfrak{D}\omega^3-2\omega-2\mathfrak{B}\omega^2+2\mathfrak{C}\omega^3-\text{etc.}}{\omega^2(1+\omega)},$$

which is reduced to this form

$$\frac{\omega^2+\mathfrak{C}\omega^2+\mathfrak{B}\omega^3+2\mathfrak{C}\omega^3-\mathfrak{D}\omega^3+\text{etc.}}{\omega^2(1+\omega)}$$

Now let $\omega=0$ and $1+\mathfrak{C}$ will result. Therefore, the value of the propounded expression in the case $x=1$ will be $=1+\mathfrak{C}$ and hence will be expressed by means of this series

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.};$$

since its sum can be exhibited neither by means of logarithms nor the circumference of the circle π, the value in question can still not be assigned by means of another method in a finite way. From these two examples the use which the differentiation of inexplicable functions can have in the doctrine of series is seen sufficiently clearly.

§386 In the method to differentiate inexplicable functions treated here we assumed that the infinitesimal terms of the series A, B, C, D, E etc. are either $=0$ or have finally vanishing differences; if both is not the case, this method cannot be used. Therefore, I will explain another method not restricted by this condition which yields the general summation of series derived from the general term and explained in more detail above [chap. V]. Therefore, let the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc. denote the Bernoulli numbers exhibited in § 122 and let this inexplicable function be propounded

$$\begin{array}{cccccccccccc} & & 1 & & 2 & & 3 & & 4 & & \cdots & & x \\ S & = & A & + & B & + & C & + & D & + & \cdots & + & X, \end{array}$$

and since we showed above (§ 130) that it will be

$$S = \int Xdx + \frac{1}{2}X + \frac{\mathfrak{A}dX}{1 \cdot 2dx} - \frac{\mathfrak{B}d^3X}{1 \cdot 2 \cdot 3 \cdot 4dx^3} + \frac{\mathfrak{C}d^5X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6dx^5} - \text{etc.},$$

it will therefore be easy to exhibit the differential of the function S; for, it will be

$$dS = Xdx + \frac{1}{2}dX + \frac{\mathfrak{A}ddX}{1 \cdot 2dx} - \frac{\mathfrak{B}d^4X}{1 \cdot 2 \cdot 3 \cdot 4dx^3} + \frac{\mathfrak{C}d^6X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6dx^5} - \text{etc.}$$

§387 But if the propounded progression is connected to the geometric series, in which case its infinitesimal terms are never reduced to constant differences and therefore the first method cannot be used, then the method treated in § 174 provides us with the solution. For, if this function is propounded

$$S = Ap + Bp^2 + Cp^3 + Dp^4 + \cdots + Xp^x,$$

find the values of the letters α, β, γ, δ etc. that it is

$$\frac{p-1}{p-e^u} = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \text{etc.},$$

having found which, as we exhibited them in § 173, it will be

$$S = \frac{p}{p-1} \cdot p^x \left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \frac{\delta d^4X}{dx^4} - \text{etc.} \right)$$

$\pm$ Constant which renders the sum $= 0$, if one puts $x = 0$, or which satisfies any other case. Therefore, having taken the differential this constant will go out of the computation and it will be

$$\begin{aligned} dS = {} & \frac{p}{p-1} \cdot p^x dx \ln p \left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \text{etc.} \right) \\ & + \frac{p}{p-1} \cdot p^x \left(dX - \frac{\alpha ddX}{dx} + \frac{\beta d^3X}{dx^2} - \frac{\gamma d^4X}{dx^3} + \text{etc.} \right) \end{aligned}$$

or

$$dS = \frac{p^{x+1}}{p-1} \left(Xdx \ln p - (\alpha \ln p - 1)dX + (\beta \ln p - \alpha)\frac{ddX}{dx} - (\gamma \ln p - \beta)\frac{d^3X}{dx^2} + \text{etc.} \right),$$

which is the differential in question of the propounded function S.

§388 But if the propounded inexplicable function consists of factors and their infinitesimal logarithms have constant differences or not, then by means of this method the differential of the function can always be exhibited. For, let

$$\begin{array}{rcccccc} & 1 & 2 & 3 & 4 & \cdots & x \\ S = & A \cdot & B \cdot & C \cdot & D & \cdots & X. \end{array}$$

Since it is

$$\ln S = \ln A + \ln B + \ln C + \ln D + \cdots + \ln X,$$

using the above method involving the Bernoulli numbers it will be

$$\ln S = \int dx \ln X + \frac{1}{2}\ln X + \frac{\mathfrak{A} d. \ln X}{1 \cdot 2dx} - \frac{\mathfrak{B} d^3. \ln X}{1 \cdot 2 \cdot 3 \cdot 4dx^3} + \text{etc.},$$

having differentiated which expression it is

$$\begin{aligned} \frac{dS}{S} &= dx \ln X + \frac{1}{2}d. \ln X + \frac{\mathfrak{A} dd. \ln X}{1 \cdot 2dx} - \frac{\mathfrak{B} d^4. \ln X}{1 \cdot 2 \cdot 3 \cdot 4dx^3} \\ &+ \frac{\mathfrak{C} d^6. \ln X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6dx^5} - \frac{\mathfrak{D} d^8. \ln X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8dx^7} + \text{etc.} \end{aligned}$$

Hence, if it was $X = x$ that it is

$$S = 1 \cdot 2 \cdot 3 \cdot 4 \cdots x,$$

after the application it will be

$$\frac{dS}{S} = dx \ln x + \frac{dx}{2x} - \frac{\mathfrak{A} dx}{2xx} + \frac{\mathfrak{B} dx}{4x^4} - \frac{\mathfrak{C} dx}{6x^6} + \frac{\mathfrak{D} dx}{8x^8} - \text{etc.},$$

which form, if x was a very large number, is applied more conveniently than the one we found before.